The rheophysics of two athermal systems:

dry granular media and non-Brownian suspensions

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Résumé :

L'écoulement de cisaillement simple à pression contrôlée constitue un paradigme des écoulements de systèmes athermiques comme les grains secs et les suspensions non Browniennes. L'introduction de deux nombres adimensionnés, inertiel et visqueux, a constitué une importante étape. Nous introduisons deux équations de base régissant le volume libre et le rapport contrainte de cisaillement sur la pression en fonction de lois de puissance des nombres inertiels et visqueux. La première loi dérive de l'équation de Boltzmann exprimée dans l'approximation de relaxation ; la deuxième loi est déduite de la probabilité pour une particule d'appartenir à un cluster transitoire de particules en contact.

Abstract :

The steady-state, pressure-controlled simple shear flow constitutes a paradigm for the study of flows of dense athermal systems such as dry grains or non-Brownian suspensions. The introduction of two dimensionless numbers called inertial and viscous was an important step. In an attempt to provide a common rheophysical framework to both systems, we relate their flow properties in the dense regime close to jamming transition to two basic constitutive laws or equations of state : both the free volume and the ratio of shear stress to particle pressure (in excess of a static friction coefficient) vary as power laws, with system-dependent exponents, of a Deborah number. The first law is derived from the Boltzmann equation expressed in the approximation of relaxation. The second law is deduced from the expression of the probability for a particle to belong to a transient cluster of particles in contact.

Mots clefs : *systèmes athermiques, rhéophysique.*

Introduction

The physics of divided matter as granular media or suspensions of grains immersed in a viscous fluid depends crucially on the size d of the grains. The critical size separating the grains subjected to thermal agitation, as Brownian motion, and those insensitive to temperature is roughly around a micrometer. The media consisting of grains whose size is greater than this critical size constitute athermal divided media. For such systems, the physics of contact between grains is of great importance.

The rheophysics of these athermal systems still lack a conceptual framework similar to the one used for statistical thermodynamics. In this paper we will focus on the rheology of two amorphous media: non Brownian suspensions and granular media of hard particles near the jamming transition. We have attempted to identify a set of variables that can be used in the description of athermal systems, such as granular or dense suspension shear flows. We support the assumption that the shear rate $|\dot{\gamma}|$, generating velocity fluctuations, defined as the second invariant of the shear rate tensor $\sqrt{\dot{\gamma}_{ij}\dot{\gamma}_{ij}}$ [1], allows to exploit the different configurations of the phase space.

The steady-state, pressure-controlled simple shear flow constitutes a paradigm for the study of flows of dense athermal systems such as dry grains or non-Brownian suspensions. The introduction of two dimensionless numbers called inertial and viscous [2, 3, 4, 5] was an important step in the understanding of such flows.

In dry granular systems, for small shear rates (at volume fraction or at pressure fixed), the flow is quasi-static with an increasing number of caged particles, with maintained friction contact, forming transient rigid clusters. Inside, the local solid fraction is larger than $\phi*$, the critical solid fraction at the jamming transition. That critical fraction is a decreasing function of the interparticle friction and can vary between .58 and .64. [6]). It is assumed that inside the clusters, the interaction is purely frictional and is described by the internal static friction $\mu_S(\mu)$ [6]. For large values of the shear rate, the flow becomes collisional dense, with an increasing number of slightly correlated free particles, with local solid fraction smaller than $\phi*$. Therefore, we consider that among the population of flowing free particles the interactions are collisional. As a consequence, the total shear stress can be written as the sum of two components: a static frictional term, proportional to the dynamic pressure $\mu_S P$ inside the rigid clusters, plus dynamic collisional or viscous terms.

In suspensions or in granular systems without friction, there is no experimental evidence of existence of such structure. For such systems we assume, in the following, that the rheology depends principally of the nearest neighbors of each particle.

1-Free volume relaxation times.

1-1 Without friction

The average structural relaxation times τ , for a viscous suspension, or for a granular are expressed as in a Kelvin-Voigt element as the ratio of hydrodynamic viscosity $\eta_H / \eta_0 \approx \phi^2/(\phi* - \phi) = \phi g_0$ with η_0 the viscosity of the suspending medium or granular viscosity $\eta_g \approx \phi\, {g_0}^2 \rho_g d^2 \dot{\gamma}$ and particles pressure.

For a suspension: $\tau = \eta_H / P^s$ and for a granular: $\tau = \eta_g / P^g$

The value of the critical fraction $\phi*$ will be specified in Section 2.

From those expressions, we define two Deborah numbers :

$$D = \tau \dot{\gamma}$$

An important step in understanding the rheology of dry and immersed granular media was the introduction of two dimensionless numbers [2,3,4,5] :

for granular $I = d\sqrt{\rho_p / P^g}\,\dot{\gamma}$ and for suspension $I_v = \eta_0 \dot{\gamma} / P^s$

In the next section we will show that:

for granular $D \approx g_0^{-1} \cong I^{2/3}$ for suspension $D \approx g_0^{-1} \cong I_v^{1/2}$

1-2 With friction

The Deborah number of a free particle with friction is:

$D_\mu = \tau \dot{\gamma} = \dfrac{\mu_d P + \eta\dot{\gamma}}{P} = \mu_d + D$ where μ_d is the dynamic friction of the flowing particles.

To conclude this section, we point out that the Deborah numbers D_μ can be interpreted as the effective friction μ^* of a free particle at a mesoscopic level.

2- State equation of athermal systems: granular media and suspensions.

In a simple shear flow, we suggest the following picture for the flow at small Deborah number $D = \dot{\gamma}\,\tau$: the flow properties at small (and zero) stress and volume fractions just below the jamming fraction ϕ^* are governed by the vicinity of the isostatic state. In granular, this is put into evidence by a correlation length $l(\phi)$ that increases on approaching the critical state at ϕ^* from below [8, 9].

We consider a granular system or a suspension sheared between two planes. A normal stress σ_{yy} is imposed .

The evolution of the volume fraction, in a shear suspension or in a shear granular medium is governed by two antagonistic processes described in the right hand side of a kinetic equation.

This equation describes dissipative systems and a tentative to show it is issued from a Boltzmann equation 'in the relaxation time approximation'[11] is developed hereafter.

The Boltzmann equation can be written:

$$\frac{Df}{Dt} = \left(\frac{\partial f}{\partial t}\right)_{collisions}$$

Here $f(\vec{v},\vec{r},t)$ is the velocity distribution function.

In the right side of the equation, the collision term can be separated into two.

We assume here that, after the cut off of the imposed shear stress, and consequently the cut of the shear rate, the effect of collisions is to restore a local equilibrium situation, described by the distribution function f_0, after a relaxation time τ:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{\partial \vec{v}}{\partial t}\cdot\frac{\partial f}{\partial \vec{v}} = \frac{f_0 - f}{\tau}$$

Here $f(\vec{v},\vec{r},t)$ is the velocity distribution function; the relaxation time τ depends only on the f-momenta, and $f_0 = \lim_{v\to 0} f(\vec{v},\vec{r})$. This limit leads to the densest configuration $\hat{\phi} = \phi_{rcp} = .64$ for frictionless particles.

The second term in the equation above is zero in a homogeneous medium ($\vec{\nabla} f = 0$), the third term

is zero if the acceleration is zero.
Restoring the external forces, we assume now the distribution function being governed by:

$$\frac{\partial f}{\partial t} = \frac{f_0 - f}{\tau} + [div(\vec{v}\, f)]_{collisions}$$

The second term on the right is equal to the divergence of the flux $\vec{v}f$ created by collisions; that latter term vanishes if the distribution is reversible during interactions. On the contrary, when it reflects the irreversible change of the distribution function, we can approximate the divergence term by: $\frac{\partial}{\partial y}(v_y f) \approx -\frac{1}{d}\dot{\gamma}\,\delta a\, f$ where δa is the irreversible shift of the particle streamline due to interaction (Fig1). After integration over the velocities, $\phi = \frac{\pi d^3}{6}\int_v f\, dv$, $\hat{\phi} = \frac{\pi d^3}{6}\int_{v\to 0} f\, dv$ (here $\hat{\phi} = \phi_{rcp}$ is the volume fraction of the maximum packing of an isotopic configuration of spherical particles without friction ($\hat{\phi} = \phi_{rcp} = .64$) we can write:

$$\frac{\partial \phi}{\partial t} = \frac{\hat{\phi} - \phi}{\tau} - (2\frac{\delta a}{d})\phi\dot{\gamma}$$

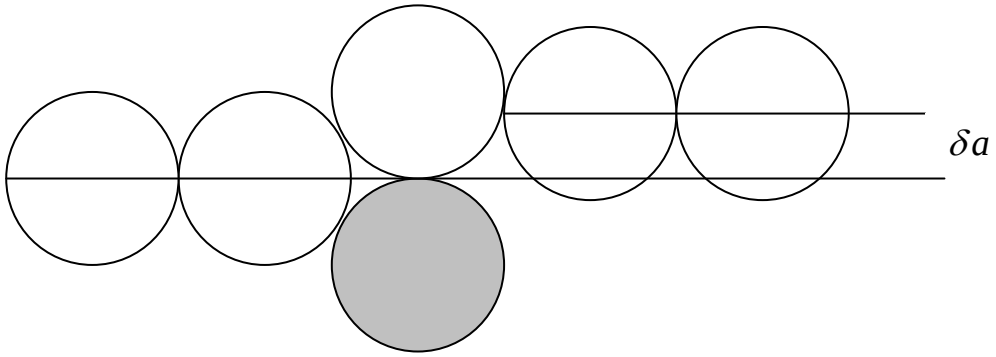

Figure 1. Schematic description of an irreversible pair interaction: δa is a mean value of the shift of the particle streamline that depends evidently on the impact parameter. The irreversibility of a pair interaction was studied in [15, 16, 17]. In dilute suspensions, the value of δa was related to the roughness of particles in suspension (15,17). In concentrated systems, the shift is due to multiple interactions. More generally, δa could be measured by following a particle in an experiment or in a computer simulation. In granular the 2D simulations in [2,3] leads to $2\frac{\delta a}{d} \approx .4$

Then we assume the evolution of the mean volume fraction ϕ, being governed by:

$$\frac{\partial \phi}{\partial t} = \frac{\hat{\phi} - \phi}{\tau} - \alpha\,\phi\dot{\gamma} \qquad (2\text{-}1)$$

The second term in the right member of the above equation reflects the frequency of irreversible interactions per volume unit.

In the equation (2-1), the packing fraction $\hat{\phi}$ is the random dense packing. Introducing the Deborah numbers of particles without friction:

$$\frac{\tau}{\phi}\frac{\partial\phi}{\partial t} = \frac{\hat{\phi}-\phi}{\phi} - \alpha\, D \text{ with } \alpha = 2\frac{\delta a}{d} \quad .$$

The imposed normal stress σ_{yy} tends to decrease the free volume, while the collisional flow $\alpha\phi\dot{\gamma}$ tends to expand the volume occupied by the grains. The latter term originates from the irreversibility of particle interactions under shear. Such time irreversibility stems from the multiple interactions and the interparticle friction. In the following sections, this parameter $\alpha < 1$ will be chosen equal to 1 in order to simplify the different expressions.

In a steady state, $\partial\phi/\partial t = 0$, and for concentrated systems, the dynamic equilibrium between the two processes provides a relationship between the volume fraction and the Deborah numbers:

$$(\hat{\phi}-\phi)/\phi = D \quad (2-2).$$

Introducing now the friction, the equation (3-2) can be rewritten:

$$(\hat{\phi}-\phi)/\phi = \mu_d + D \quad (2-2-a)$$

By decreasing the numbers D down to zero, the volume fraction tends to the critical fraction $\phi^*(\mu)$ defined by: $(\hat{\phi}-\phi^*)/\phi^* = \mu_c$, and it is easy to show that:

$$1/g_0 = (\phi^*-\phi)/\phi \cong D \quad (2-2,b)$$

The fraction $\phi^* = \hat{\phi}/(1+\alpha\mu_d)$, to be compared to the results in [6], is the limit value $\phi^* \approx .58$ experimentally obtained by Boyer et al. [10].

Thanks to the expression of g_0, and to the equation (2-2,b)), we can formulate those Deborah numbers as a function of the control parameters D_0. For suspensions, the control parameter $D_0^s = \eta_0\dot{\gamma}/P^s$ corresponds to the viscous number I_v introduced by Cassar and Al.[5]. In the case of granular systems, D_0^g corresponds to the inertial number I introduced by Da Cruz and Chevoir [6,7]: $I = d\sqrt{\rho_p/P^g}\,\dot{\gamma}$.

We can therefore distinguish two cases:

A- For suspensions, we notice that equation (2-2 b) can be written: $1/g_0 = D = (\eta_H/\eta_0)\,\eta_0\dot{\gamma}/P \approx g_0\, I_v$, then $D^s = (I_v)^{0,5}$, and finally $(\phi^*-\phi)/\phi = (I_v)^{0,5}$ Recently, Boyer and al. [10], have proposed an empirical expression identical to this equation to fit their experimental results.

B- In the case of a three-dimensional granular media with friction, the equation of state can be written , $g_0^{-1} = I^{2/3}$; such an analytical expression of the state equation accounts well for the numerical results of Hatano [18,19] $\phi^*-\phi = .11\, I^{.56}$; the numerical results of Peyneau and Roux [20] are also well fitted by a power law.

3- Constitutive equations

In granular, the effective friction μ^*, ratio of the shear stress and the pressure appears as a combination of frictional and collisional terms, each term being weighted by the function $f(D)$ that represents the proportion of caged particles in transient clusters. We notice that

$f \cong 1/(1+kD)$ for $kD << 1$, an expression proposed in [14] to fit results of numerical simulations.

Both components of the shear stress give the total shear stress:

$$\tau = \mu_S P\ f(D) + (\mu_d + D)\, P\,(1 - f(D)) \quad ;\ f = \exp(-kD)$$

$$\mu^* = \mu_S\ f(D) + (\mu_d + D)\,(1 - f(D))$$

For granular, $D = I^{2/3}$ and $\mu_d(\mu)$ is a growing function of the interparticle friction μ [10].

Up to now, an experimental evidence of dynamical heterogeneities in athermal suspensions is yet an open problem. It follows that the effective friction is the same at the mesoscale and at the macroscale.

Then, for suspension, $\mu^* = \mu_S + D \quad D = I_v^{1/2}$

Conclusion

We have shown that the rheology of very different athermal systems such as dry granular media, immersed granular media, suspensions of hard particles, can be described by equations whose general structure is the same for all these systems. The specificity of each system is reflected in the specific expression of the Deborah numbers.

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